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ISOSCELES TRIANGLES ON THE SIDES OF A TRIANGLE

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Original scientific paper

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ABSTRACT

Famous construction of Fermat-Toricelly point of a triangle leads to the question is there a similar way to construct other isogonic centers of a triangle in a similar way. For a purpose we remember that Fermat-Torricelli point of a triangle ΔABC is obtained by constructing equilateral triangles outwardly on the sides AB,BC and CA. If we denote thirth vertices of those triangles by C_1,A_1 and B_1 respectively, then the lines AA_1,BB_1 and CC_1 concurr at the Fermat-Torricelli point of a triangle ΔABC (Van Lamoen, 2003). In this work we present the condition for the concurrence, of the lines AA_1,BB_1 and C_1, where C_1,A_1 and B_1 are the vertices of an isosceles triangles constructed on the sides AB,BC and CA (not necessarily outwordly) of a triangle ΔABC. The angles at this work are strictly positive directed so we recommend the reader to pay attention to this fact.

Keywords: *Ceva, Menelaus, Stewartes, cevian, concurrency, collinearity, Fermat, Torricelly*

INTRODUCTION

Leading idea for this work was Napoleon Triangles and Kiepert Perspectors, submitted by Floor van Lamoen (2003) to Forum Geometricorum in which the complex numbers are used to show the existance and the construction of Fermat-Toricelly point. Observing the hystorical facts we can se the Fermats-Toricelly point is one of the extremal points of a triangle, same as the centroid is. Namely if the point O is constructed in the plane of a triangle ΔABC then the sum AO+BO+CO is minimal if and onl if O coinsides with Fermat-Toricellis point of a triangle ΔABC (Prasolov, 2001). Later as a special case we will see this one leads to the condition $\measuredangle AOC = \measuredangle BOA = \measuredangle COB = \frac{2\pi}{3}$. The sum $AO^2+BO^2+CO^2$ is minimal if and only if O coinsides with the centroid of a triangle ΔABC (Altshiller-Court, 2007). One can ask the quaestion when the sum $AO^3+BO^3+CO^3$ is minimal, or some other questions. The theorem we present shows that any point in the plane of a triangle can be constructed using an issoceles triangles and certain condition.

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MAIN THEOREM

Theorem 1. In a nondegenerated triangle ∡ABC, ∡CAB=α, ∡ABC=β, ∡BCA=γ. Let the points A_1,B_1 and C_1 lie in the plane of a triangle such that ∡ACB[1]= ∡B_1AC=φ ∡BAC_1= ∡C_1 BA=ω and ∡CBA_1= ∡A_1 CB=δ. The lines AA_1, BB_1 and CC_1 are concurrent or parallel if and only if

$$\sin(\omega+\alpha)\cdot\sin(\varphi+\gamma)\cdot\sin(\beta+\delta)=\sin(\omega+\beta)\cdot\sin(\varphi+\alpha)\cdot(\delta+\gamma)\Leftrightarrow$$
$$\sin(\varphi-\omega)\cdot\cos(2\alpha-\delta)+\sin(\omega-\delta)\cdot\cos(2\beta-\varphi)+\sin(\delta-\varphi)\cdot\cos(2\gamma-\omega)=0$$

Proof:

Let us consider the case

$$\sin(\omega+\alpha)\cdot\sin(\varphi+\gamma)\cdot\sin(\beta+\delta)\cdot\sin(\omega+\beta)\cdot\sin(\varphi+\alpha)\cdot(\delta+\gamma)=0$$

Let $\sin(\omega+\alpha)=0$. Since a triangle ΔABC is nondegenerated, thus $\omega+\alpha\neq 0$ so we have $\omega+\alpha\in\{\pi,2\pi\}$.

Let ω+α=π, then ∡BAC_1+ ∡CAB=π, which means C_1 lies on the extension of the line CA such that A is between the points C and C_1. Since ∡BAC_1= ∡C_1 BA=ω=π-α then we have $2(\pi-\alpha)<\pi\Rightarrow\alpha>\frac{\pi}{2}$.

Let AA_1 and CC_1 meet at A then BB_1 also contains the point A. Thus B_1 lies on the line AB. Since ∡ACB_1= ∡B_1 AC, and $\alpha>\frac{\pi}{2}$, then A is between the points B and B_1. Now we have ∡ACB_1= ∡B_1 AC=φ=ω so we have

$$\varphi+\alpha=\pi\Rightarrow\sin(\varphi+\alpha)=0\Rightarrow$$
$$\sin(\omega+\alpha)\cdot\sin(\varphi+\gamma)\cdot\sin(\beta+\delta)=0=\sin(\omega+\beta)\cdot\sin(\varphi+\alpha)\cdot(\delta+\gamma)$$

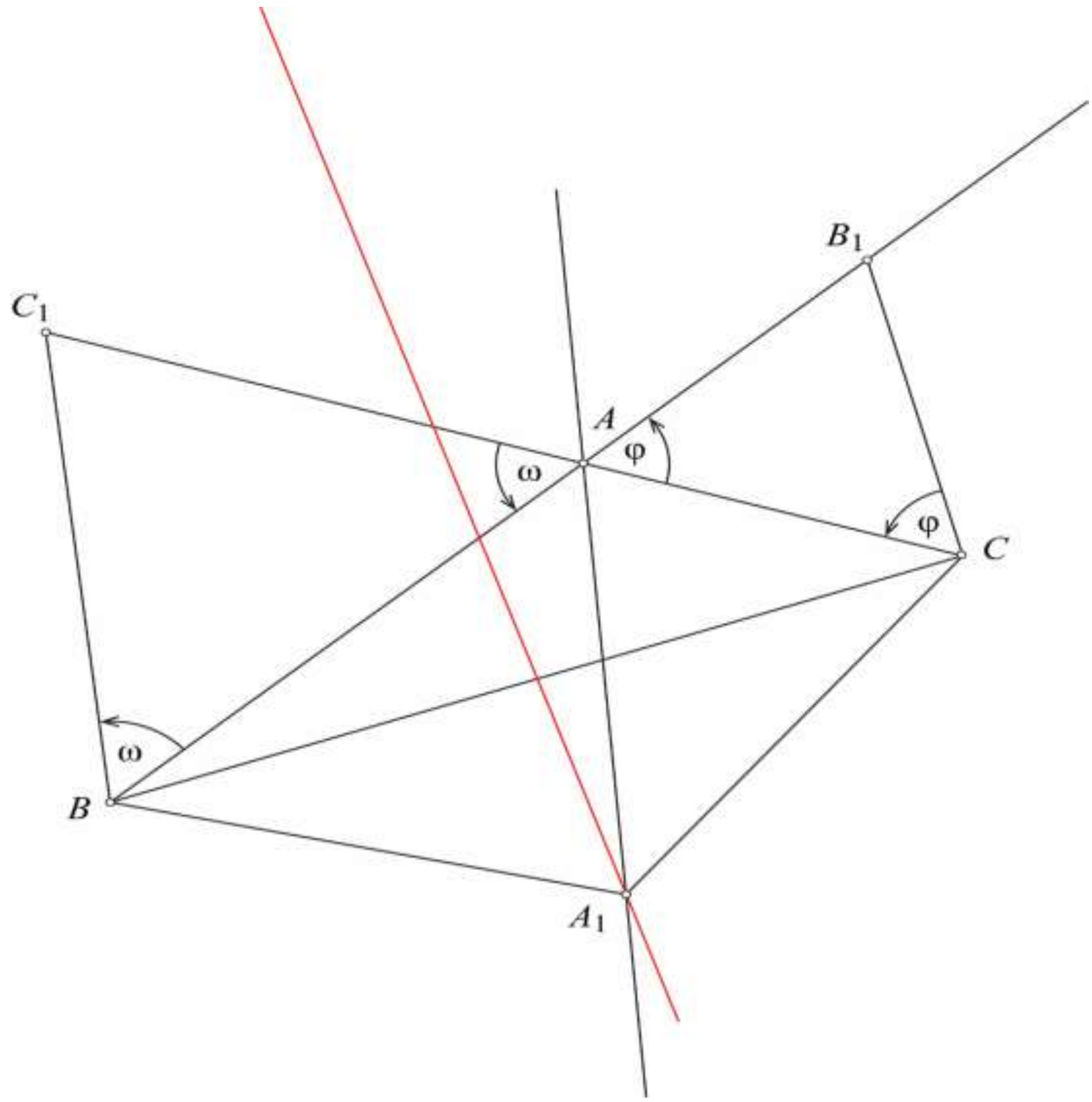

Figure 1.

Then A_1 is any point on the bisector of the segment BC. Let AA_1 be parallel to CC_1, then A_1 lies on the line CA. If BB_1 is also parallel to CC_1 then B_1 is an intersection point of a line through B parallel to CA and the bisector of the segment CA.

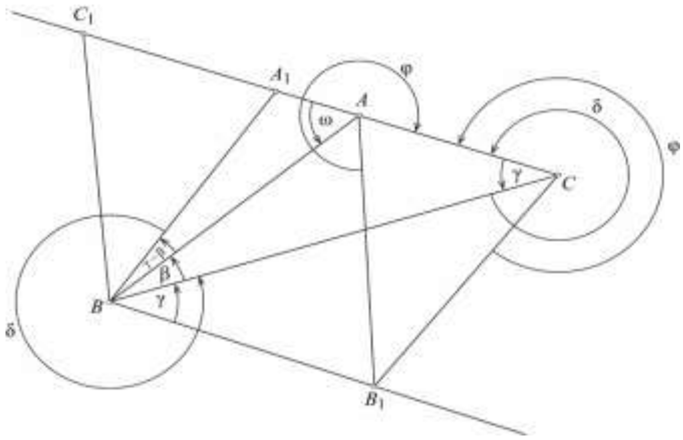

Figure 2.

But then we have

$$\measuredangle A_1CB + \measuredangle BCA = 2\pi \Rightarrow \delta + \gamma = 2\pi \Rightarrow \sin(\delta + \gamma) = 0 \Rightarrow$$

$$\sin(\omega + \alpha) \cdot \sin(\varphi + \gamma) \cdot \sin(\beta + \delta) = 0 = \sin(\omega + \beta) \cdot \sin(\varphi + \alpha) \cdot (\delta + \gamma)$$

Let $\omega+\alpha=2\pi$ so the point C_1 is on the line CA such that C and C_1 are on the same side of the point A. But then we have

$$\measuredangle BAC_1 = \measuredangle C_1BA \Rightarrow 2\pi - \measuredangle BAC_1 = 2\pi - \measuredangle C_1BA \Rightarrow$$

$\measuredangle C_1AB = \measuredangle ABC_1 \Rightarrow \alpha < \frac{\pi}{2}$. Let CC_1 and AA_1 meet at the point A. Then BB_1 contains the point A only if B_1 is on the line BA. Then we have

$$\measuredangle B_1AC = \measuredangle BAC_1 \Rightarrow \varphi = \omega \Rightarrow \varphi + \alpha = 2\pi \Rightarrow \sin(\varphi + \alpha) = 0 \Rightarrow$$

$$\sin(\omega + \alpha) \cdot \sin(\varphi + \gamma) \cdot \sin(\beta + \delta) = 0 = \sin(\omega + \beta) \cdot \sin(\varphi + \alpha) \cdot (\delta + \gamma)$$

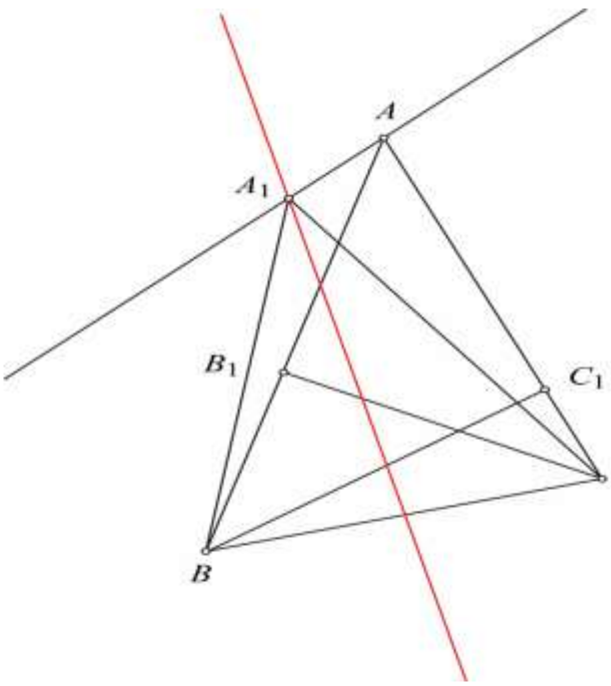

Figure 3.

Let AA_1 be parallel to the line CC_1, Then BB_1 parallel to them so B_1 is an interesection point of the line throught B parallel to AC and the bisector of the segment AC. But then

$$\measuredangle A_1CB + \measuredangle BCA = 2\pi \Rightarrow \delta + \gamma = 2\pi \Rightarrow \sin(\delta+\gamma) = 0 \Rightarrow$$

$$\sin(\omega+\alpha)\cdot\sin(\varphi+\gamma)\cdot\sin(\beta+\delta) = 0 = \sin(\omega+\beta)\cdot\sin(\varphi+\alpha)\cdot(\delta+\gamma)$$

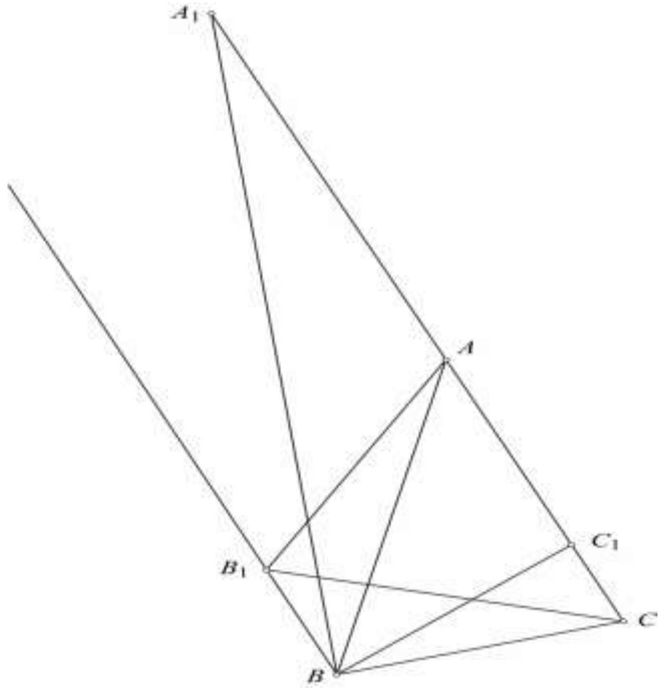

Figure 4.

Similarly we reconsider the cases remained from the equation

$$\sin(\omega+\alpha)\cdot\sin(\varphi+\gamma)\cdot\sin(\beta+\delta)\cdot\sin(\omega+\beta)\cdot\sin(\varphi+\alpha)\cdot(\delta+\gamma)=0$$

As we can notice, the intersection points of the lines AA_1, BB_1 and CC_1 are the triangle vertices A,B and C or the point at infinity.

Suppose now that

$$\sin(\omega+\alpha)\cdot\sin(\varphi+\gamma)\cdot\sin(\beta+\delta)\cdot\sin(\omega+\beta)\cdot\sin(\varphi+\alpha)\cdot(\delta+\gamma) \neq 0$$

Consider the points A and A_1 being from distinct sides of a line BC. Let the line AA_1 meets the line BC at the point A'. Let the line through A_1 parallel to BC meets lines AB and AC at the points D and E respectively. From the similarity ΔABC~ΔADE we have

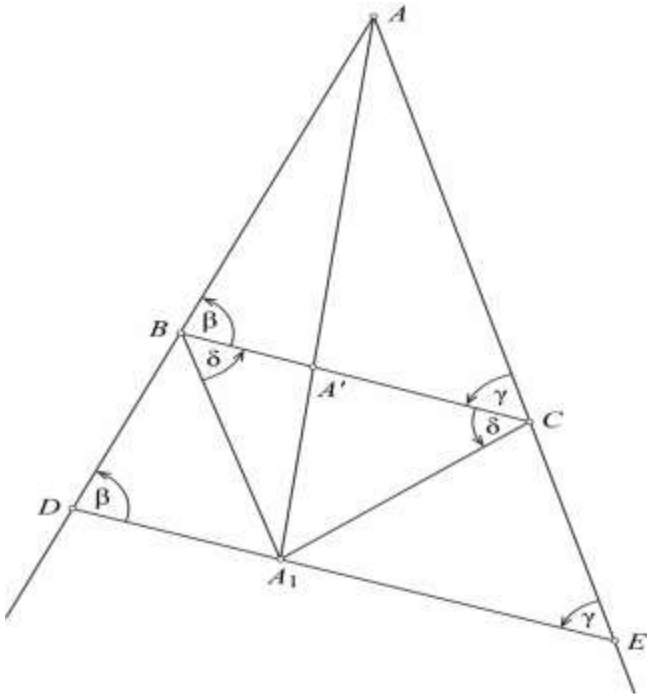

Figure 5.

$$\frac{A_1D}{A_1E} = \frac{BA'}{A'C}$$

From the sine theorem we have

$$A_1D = \frac{A_1B}{sin\beta} \cdot \sin(\beta + \delta)$$

$$A_1E = \frac{A_1C}{sin\gamma} \cdot \sin(\gamma + \delta)$$

Dividing we get

$$\frac{BA'}{A'C} = \frac{sin\gamma}{sin\beta} \cdot \frac{\sin(\beta + \delta)}{\sin(\gamma + \delta)}$$

Let now A and A_1 be from the same side of the line BC Let the line AA_1 meets the line BC at the point A'. Let the line through A_1 parallel to BC meets lines AB and AC at the points D and E respectively. From the similarity ΔABC~ΔADE we have

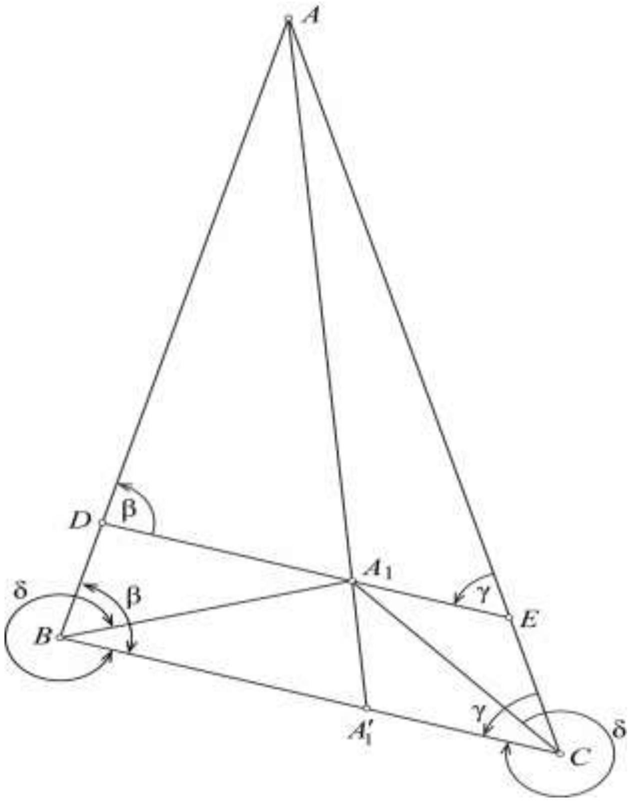

Figure 6.

$$\frac{A_1D}{A_1E} = \frac{BA'}{A'C}$$

From the sine theorem we have

$$A_1D = \frac{A_1B}{sin\beta} \cdot \sin(\beta + \delta - 2\pi)$$

$$A_1E = \frac{A_1C}{sin\gamma} \cdot \sin(\gamma + \delta - 2\pi)$$

Dividing we get

$$\frac{BA'}{A'C} = \frac{sin\gamma}{sin\beta} \cdot \frac{\sin(\beta + \delta)}{\sin(\gamma + \delta)}$$

So in any case we have

$$\frac{BA'}{A'C} = \frac{sin\gamma}{sin\beta} \cdot \frac{\sin(\beta + \delta)}{\sin(\gamma + \delta)}$$

Let us define the points B' and C' similarly. By Cevas theorem (Kedlaya, 1999) the lines AA_1,BB_1 and CC_1 meet at the point or are parallel if and only if

$$\frac{BA'}{A'C}\cdot\frac{CB'}{B'A}\cdot\frac{AC'}{C'B}=1\Leftrightarrow$$

$$\frac{\sin(\beta+\delta)\cdot\sin(\varphi+\gamma)\cdot\sin(\omega+\alpha)}{\sin(\gamma+\delta)\cdot\sin(\varphi+\alpha)\cdot\sin(\omega+\beta)}=1\Leftrightarrow$$

$$\sin(\varphi-\omega)\cdot\cos(2\alpha-\delta)+\sin(\omega-\delta)\cdot\cos(2\beta-\varphi)+\sin(\delta-\varphi)\cdot\cos(2\gamma-\omega)=0$$

CONSEQUENCES WHEN δ=φ=ω

Corollary 1. On the sides of a nondegenerated triangle ΔABC are constructed regular n-gons outwardly, $AC_2\dots C_{n-1}B$,$BA_2\dots A_{n-1}C$ and $CB_2\dots B_{n-1}A$.Let C_1,A_1 and B_1 be the centers of those polygones respectively. Then the lines AA_1,BB_1 and CC_1 concurr.

Proof:
Since the triangles ΔAC_1B,ΔBA_1C and ΔCB_1A are an issoceles triangles constructed on the sides of nondegenerated triangle ΔABC and $\delta=\varphi=\omega=\frac{n-2}{2n}\pi$, applying the theorem 1 in its second equivalent form directly implies the claim.

Corollary 2. On the sides of a nondegenerated triangle ΔABC are constructed regular 2n+1-gons outwardly, $AC_2\dots C_{2n}B$, $BA_2\dots A_{2n}C$ and $CB_2\dots B_{2n}A$..Then the lines AA_{n+1}, BB_{n+1} and CC_{n+1} concurr.

Proof:
Since the triangles $\Delta AC_{n+1}B$, $\Delta BA_{n+1}C$ and $\Delta CB_{n+1}A$ are an issoceles triangles constructed on the sides of nondegenerated triangle ΔABC and $\delta=\varphi=\omega=\frac{n-1}{2n}\pi$ applying the theorem 1 in its second equivalent form directly implies the claim.

Corollary 3. On the sides of a nondegenerated triangle ΔABC are constructed regular 2n-gons outwardly, $AC_2\dots C_{2n-1}B$, $BA_2\dots A_{2n-1}C$ and $CB_2\dots B_{2n-1}$.Let C_1, A_1 and B_1 be themidpoints of the sides A_nA_{n+1}, B_nB_{n+1} and C_nC_{n+1} respectively. Then the lines AA_1,BB_1 and CC_1 concurr.

Proof:
Since the triangles ΔAC_1B, ΔBA_1C and ΔCB_1A are an issoceles triangles constructed on the sides of nondegenerated triangle ΔABC and δ=φ=ω, applying the theorem 1 in its second equivalent form directly implies the claim.
The corollaries obviously hold when the polygons are constructed inwardly.
Let us just draw the case when all the triangles are outwards

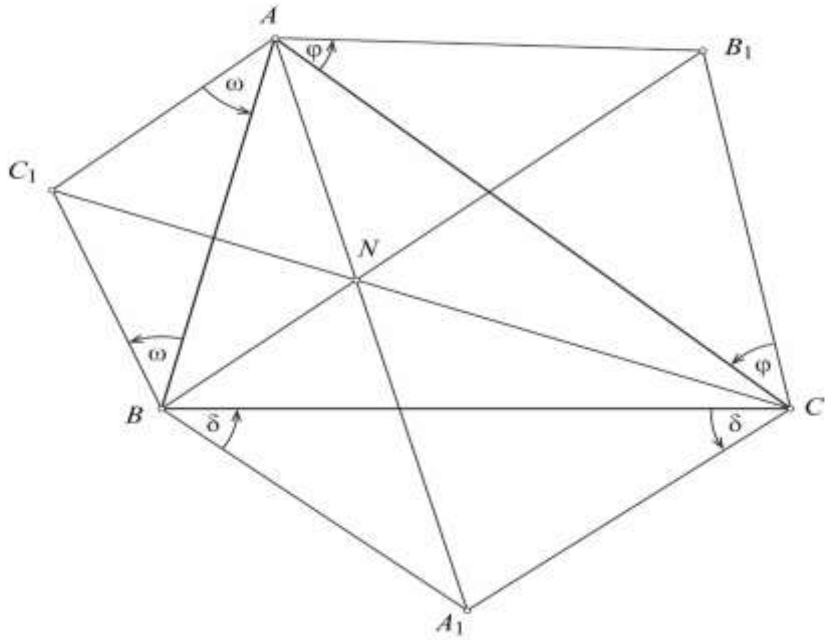

Figure 7.

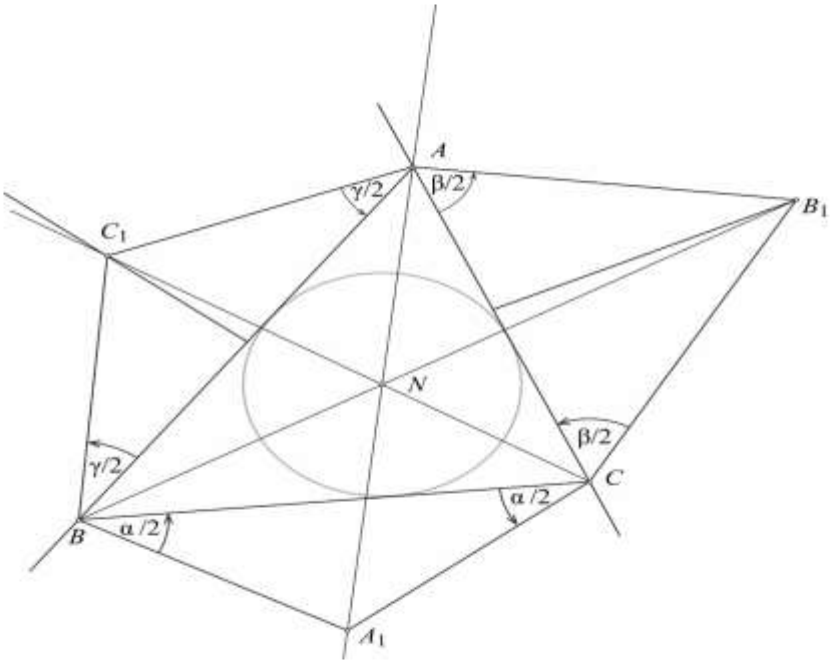

Figure 8.

This is the special case when the lies meet at the incenter.

Then below is the special case when the lines meet at Fermat-Toricelly point (Prasolov, 2001)

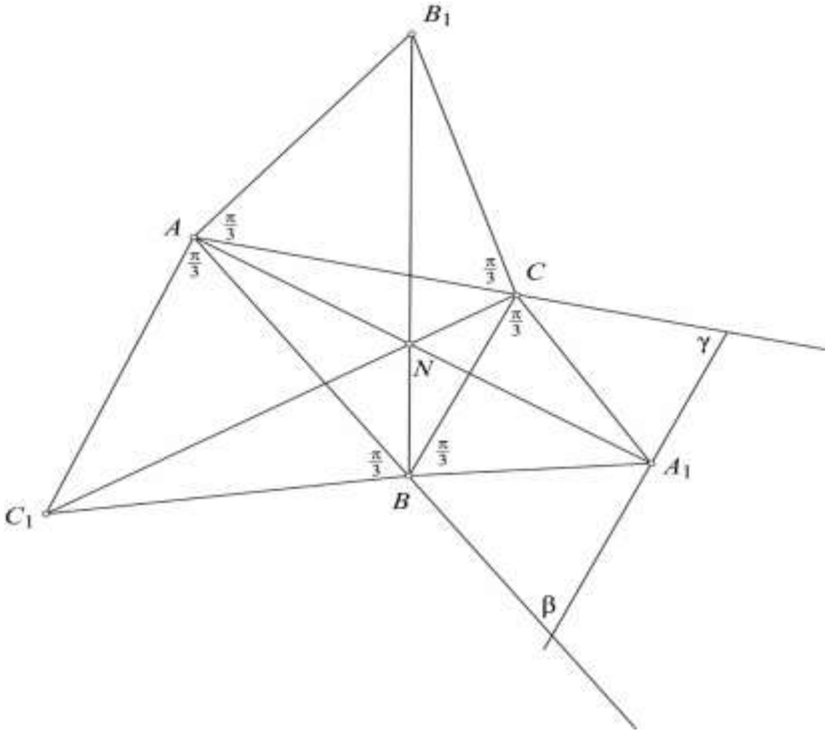

Figure 9.

CONCLUSION

Any point in the plane of nondegenerated triangle can be constructed using this method except the points belonging to the altitudes of the triangle excluding its vertices which can be constructed. This fact is obvious, any point can be connected to the vertices of a triangle, thus forming a line. The intersections of those three lines with the bisectors of the sides opposing to the vertices respectively, form three vertices of required issoceles triangles, which is not the case only if the one of the points lie on the line containing the altitude. Then connecting this point to the vertex form a line parallel to the bisector of the opposing side, hence these two lines dont meet. So there is no required issoceles triangle. Also we can see that if the point is constructible this way, then the way of construction is unique.

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